A CLOSED-FORM EXPRESSION FOR $\zeta(2n+1)$ REVEALS A SELF-RECURSIVE FUNCTION

MICHAEL A. IDOWU

SIMBIOS CENTRE,
UNIVERSITY OF ABERTAY,
DUNDEE DD1 1HG, UK
M.IDOWU@ABERTAY.AC.UK

Abstract. Euler discovered a formula for expressing the value of the Riemann zeta function for all even positive integer arguments. A closed-form expression for the Riemann zeta function for all odd integer arguments, based on the values of the Dirichlet beta function, euler numbers and π, reveals a new evidence about the self-recursive nature of Riemann zeta function at odd integers. We demonstrate for the first time that the Riemann zeta function at odd integers always produces a recurrence relation that is self-recursive.

Keywords: Riemann zeta function, Dirichlet beta function, polygamma function, closed-form expressions, odd integer arguments

1. Introduction

Nearly all number theorists have sought for a closed-form expression for $\zeta(2n+1)$; n being a positive integer number. Our investigation into this open problem has uncovered a self-recursive function intrinsic in $\zeta(2n+1)$. We develop and present a new method for deriving a closed-form expression for $\zeta(2n+1)$ which is based on the values of the Dirichlet beta function at $2n+1$ and Euler numbers at $2n$. This new method may be regarded as a general formula for finding a closed-form self-recursive expression for $\zeta(2n+1)$.

Here we demonstrate how to obtain a closed-form expression for $\zeta(2n+1)$ with examples on $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$. Each result produced is an exact representation, but the resultant expression is self-recursive.

1.1. **Riemann zeta and Dirichlet beta functions.** The Dirichlet beta function [1] is defined as

$$\beta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \dots \tag{1.1}$$

1.1 implies

$$\beta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^s} = (1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots) - 2(\frac{1}{3^s} + \frac{1}{7^s} + \frac{1}{11^s} + \dots); \tag{1.2}$$

$Re(s) > 0$. Therefore,

$$\beta(s) = \frac{2^s - 1}{2^s}\zeta(s) - 2\sum_{k=1}^{\infty} \frac{1}{(4k-1)^s} \tag{1.3}$$

$$\beta(s) = \frac{2^s - 1}{2^s}\zeta(s) - \frac{2}{2^s.2^s}\sum_{k=0}^{\infty} \frac{1}{(k+\frac{3}{4})^s}. \tag{1.4}$$

The following equations are then obtained according to [2]

$$\zeta(s) = \frac{2^s}{2^s - 1}\beta(s) + (-1)^s \frac{2}{2^s(2^s-1)} \frac{1}{\Gamma(s)} \psi^{(s-1)}(\frac{3}{4}); \tag{1.5}$$

$$\zeta(2s+1) = \frac{2^{2s+1}}{2^{2s+1}-1}\beta(2s+1) - \frac{2}{2^{2s+1}(2^{2s+1}-1)} \frac{1}{\Gamma(2s+1)} \psi^{(2s)}(\frac{3}{4}), \tag{1.6}$$

where the polygamma function $\psi^{(s-1)}(x)$ is defined as

$$\psi^{(s-1)}(x) = \frac{d^{s-1}}{dx^{s-1}}\psi(x) = \frac{d^s}{dx^s}\ln\Gamma(x). \tag{1.7}$$

2. Closed-form expressions for $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$

Equation 1.6 may be used to find the closed-form expressions for $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$.

2.1. **A closed-form expression for the Riemann zeta of 3.** From 1.6 we derive

$$\zeta(3) = \frac{2^3}{2^3-1}\beta(3) - \frac{2}{2^3(2^3-1)}\frac{1}{\Gamma(3)}\psi''(\frac{3}{4}) = \frac{8}{7}\beta(3) - \frac{2}{8(7)}\frac{1}{2!}(2(\mathbf{1})\pi^3 - 2(\mathbf{28})\zeta(3))$$

$$\rightarrow \zeta(3) = \frac{8}{7}\beta(3) - \frac{1}{56}(2\pi^3 - 56\zeta(3)) \tag{2.1}$$

2.2. **A closed-form expression for the Riemann zeta of 5.** Similarly:

$$\zeta(5) = \frac{2^5}{2^5-1}\beta(5) - \frac{2}{2^5(2^5-1)}\frac{1}{\Gamma(5)}\psi''''(\frac{3}{4}) = \frac{32}{31}\beta(5) - \frac{2}{32(31)}\frac{1}{4!}(8(\mathbf{5})\pi^5 - 8(\mathbf{1488})\zeta(5))$$

$$\rightarrow \zeta(5) = \frac{32}{31}\beta(5) - \frac{2}{32(31)}\frac{1}{4!}(40\pi^5 - 11904\zeta(5)) \tag{2.2}$$

2.3. **A closed-form expression for the Riemann zeta of 7.**

$$\zeta(7) = \frac{2^7}{2^7-1}\beta(7) - \frac{2}{2^7(2^7-1)}\frac{1}{\Gamma(7)}\psi^{(6)}(\frac{3}{4}) = \frac{128}{127}\beta(5) - \frac{2}{128(127)}\frac{1}{6!}(32(\mathbf{61})\pi^7 - 32(\mathbf{182880})\zeta(7))$$

$$\rightarrow \zeta(7) = \frac{128}{127}\beta(7) - \frac{2}{128(127)}\frac{1}{6!}(1952\pi^7 - 5852160\zeta(7)) \tag{2.3}$$

2.4. **A closed-form expression for the Riemann zeta of 9.**

$$\zeta(9) = \frac{2^9}{2^9-1}\beta(9) - \frac{2}{2^9(2^9-1)}\frac{1}{\Gamma(9)}\psi^{(8)}(\frac{3}{4}) = \frac{512}{511}\beta(9) - \frac{2}{512(511)}\frac{1}{8!}(128(\mathbf{1385})\pi^9 - 128(\mathbf{41207040})\zeta(9))$$

$$\rightarrow \zeta(9) = \frac{512}{511}\beta(9) - \frac{2}{512(511)}\frac{1}{8!}(177280\pi^9 - 5274501120\zeta(9)) \tag{2.4}$$

2.5. **A closed-form expression for the Riemann zeta of 11.**

$$\zeta(11) = \frac{2^{11}}{2^{11}-1}\beta(11) - \frac{2}{2^{11}(2^{11}-1)}\frac{1}{\Gamma(11)}\psi^{(10)}(\frac{3}{4})$$

$$\rightarrow \zeta(11) = \frac{2048}{2047}\beta(11) - \frac{2}{2048(2047)}\frac{1}{10!}(512(\mathbf{50521})\pi^{11} - 512(\mathbf{14856307200})\zeta(11))$$

(2.5)

$$\rightarrow \zeta(11) = \frac{2048}{2047}\beta(11) - \frac{2}{2048(2047)}\frac{1}{10!}(25866752\pi^{11} - 7606429286400\zeta(11))$$

3. The general (closed-form expression) formula for $\zeta(2s+1)$

The results obtained in the previous sections indicate the following general formula for obtaining representing $\zeta(2s+1)$:

(3.1)

$$\zeta(2s+1) = \frac{2^{2s+1}}{(2^{2s+1}-1)}\beta(2s+1) - 2\frac{\left(2^{2s-1}\mid \mathbf{E_{2s}} \mid \pi^{2s+1} - 2^{2s-1}\mathbf{2}(\mathbf{2^{2s+1}}-\mathbf{1})\mathbf{\Gamma}(\mathbf{2s}+\mathbf{1})\zeta(2s+1)\right)}{2^{2s+1}(2^{2s+1}-1)\Gamma(2s+1)}$$

; s is an integer. The modulus $\mid E_{2s} \mid$ is the absolute value of an even-indexed Euler number E_{2s}. The implication of 3.1 is

$$2^{2s+1}(2^{2s+1})\Gamma(2s+1)\beta(2s+1) = 2^{2s}\mid \mathbf{E_{2s}} \mid \pi^{2s+1} \tag{3.2}$$

as expected. Hence,

(3.3)

$$\zeta(2s+1)=\frac{2^{2s+1}}{(2^{2s+1}-1)}\beta(2s+1)-2\frac{\left(2^{2s-1}\frac{\mathbf{E_{2s}}}{\mathbf{i}}(\pi\mathbf{i})^{2s+1}-2^{2s-1}\mathbf{2(2^{2s+1}-1)\Gamma(2s+1)}\zeta(2s+1)\right)}{2^{2s+1}(2^{2s+1}-1)\Gamma(2s+1)}$$

$\rightarrow$

(3.4)

$$\zeta(2s+1)=\frac{2^{2s+1}}{(2^{2s+1}-1)}\beta(2s+1)+\frac{\left(2^{2s-1}\mathbf{E_{2s}}(\pi i)^{2s+1}-2^{2s-1}\mathbf{2(2^{2s+1}-1)\Gamma(2s+1)}\zeta(2s+1)\mathbf{i}\right)}{2^{2s+1}(2^{2s+1}-1)\Gamma(2s+1)}\mathbf{2i}$$

$\rightarrow$

$$\zeta(2s+1)=\frac{2^{2s+1}}{(2^{2s+1}-1)}\beta(2s+1)+\left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}-\zeta(2s+1)i\right]i \tag{3.5}$$

(3.6)

$$\frac{(2^{2s+1}-1)}{2^{2s+1}}\zeta(2s+1)=\beta(2s+1)+\frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}-\zeta(2s+1)i\right]i$$

(3.7)

$$\frac{(2^{2s+1}-1)}{2^{2s+1}}\zeta(2s+1)=\beta(2s+1)+\frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\zeta(2s+1)+\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]$$

4. Series representations of $\zeta(2s+1)$ and $\beta(2s+1)$

Combining 1.3 and 3.7 we deduce

$$\frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\zeta(2s+1)+\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]=2\sum_{k=1}^{\infty}\frac{1}{(4k-1)^{2s+1}} \tag{4.1}$$

and

$$\frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\zeta(2s+1)-\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]=2\sum_{k=0}^{\infty}\frac{1}{(4k+1)^{2s+1}} \tag{4.2}$$

because

$$\frac{(2^{2s+1}-1)}{2^{2s+1}}\zeta(2s+1)=\sum_{k=0}^{\infty}\frac{1}{(4k+1)^{2s+1}}+\sum_{k=1}^{\infty}\frac{1}{(4k-1)^{2s+1}}. \tag{4.3}$$

We find

(4.4)

$$\beta(2s+1)=-\frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]=\sum_{k=0}^{\infty}\frac{1}{(4k+1)^{2s+1}}-\sum_{k=1}^{\infty}\frac{1}{(4k-1)^{2s+1}};$$

(4.5)

$$\frac{2^{2s+1}}{2^{2s+1}-1}\beta(2s+1)=-\left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]=\frac{2^{2s+1}}{2^{2s+1}-1}\left(\sum_{k=0}^{\infty}\frac{1}{(4k+1)^{2s+1}}-\sum_{k=1}^{\infty}\frac{1}{(4k-1)^{2s+1}}\right);$$

due to the following identities:

$$\zeta(2s+1)=\frac{2^{2s+1}}{2^{2s+1}-1}\left(\sum_{k=0}^{\infty}\frac{1}{(4k+1)^{2s+1}}+\sum_{k=1}^{\infty}\frac{1}{(4k-1)^{2s+1}}\right); \tag{4.6}$$

$$\zeta(2s+1)+\frac{2^{2s+1}}{2^{2s+1}-1}\beta(2s+1)=\frac{2^{2s+1}}{2^{2s+1}-1}\sum_{k=0}^{\infty}\frac{2}{(4k+1)^{2s+1}}; \tag{4.7}$$

$$\zeta(2s+1)-\frac{2^{2s+1}}{2^{2s+1}-1}\beta(2s+1)=\frac{2^{2s+1}}{2^{2s+1}-1}\sum_{k=1}^{\infty}\frac{2}{(4k-1)^{2s+1}}. \tag{4.8}$$

5. Summary of results

We confirm the following identities to be valid:

$$\zeta(2s) = (-1)^{2s}(\frac{(\psi^{(2s-1)}(\frac{1}{4}) + \psi^{(2s-1)}(\frac{3}{4}))}{2^{2s}(2^{2s}-1)})\frac{1}{\Gamma(2s)} = \frac{\pi\frac{d^{(2s-1)}}{dz^{(2s-1)}}cot(\pi z)\mid_{z\to\frac{1}{4}}}{2^{2s}(2^{2s}-1)\Gamma(2s)}; \tag{5.1}$$

(5.2)

$$\frac{2^{2s+1}}{2^{2s+1}-1}\beta(2s+1) = (-1)^{2s+1}(\frac{(\psi^{(2s)}(\frac{1}{4}) - \psi^{(2s)}(\frac{3}{4}))}{2^{2s+1}(2^{2s+1}-1)})\frac{1}{\Gamma(2s+1)} = \frac{\pi\frac{d^{(2s)}}{dz^{(2s)}}cot(\pi z)\mid_{z\to\frac{1}{4}}}{2^{2s+1}(2^{2s+1}-1)\Gamma(2s+1)};$$

(5.3)

$$\beta(2s+1) = (-1)^{2s+1}(\frac{(\psi^{(2s)}(\frac{1}{4}) - \psi^{(2s)}(\frac{3}{4}))}{2^{2s+1}(2^{2s+1})})\frac{1}{\Gamma(2s+1)} = \frac{\pi\frac{d^{(2s)}}{dz^{(2s)}}cot(\pi z)\mid_{z\to\frac{1}{4}}}{2^{2s+1}(2^{2s+1})\Gamma(2s+1)};$$

$$E_{2s} = (-1)^{2s+1}(\frac{(\psi^{(2s)}(\frac{1}{4}) - \psi^{(2s)}(\frac{3}{4}))}{(2\pi i)^{2s+1}}).2i = \frac{\pi\frac{d^{(2s)}}{dz^{(2s)}}cot(\pi z)\mid_{z\to\frac{1}{4}}}{(2\pi i)^{2s+1}}.2i; \tag{5.4}$$

$$\zeta(2s+1) = (-1)^{2s+1}(\frac{(\psi^{(2s)}(\frac{1}{4}) + \psi^{(2s)}(\frac{3}{4}))}{2^{2s+1}(2^{2s+1}-1)})\frac{1}{\Gamma(2s+1)} \tag{5.5}$$

according to [2]and [3].

6. Conclusion

The following identities are derived:

$$\zeta(2s+1) = \left[\frac{2^{2s+1}}{2^{2s+1}-1}\sum_{k=0}^{\infty}\frac{2}{(4k+1)^{2s+1}}\right] - \left[\frac{E_{2s}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]; \tag{6.1}$$

$$\zeta(2s+1) = \left[\frac{2^{2s+1}}{2^{2s+1}-1}\sum_{k=1}^{\infty}\frac{2}{(4k-1)^{2s+1}}\right] + \left[\frac{E_{2s}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right] \tag{6.2}$$

from the general formula and relation

$$\frac{(2^{2s+1}-1)}{2^{2s+1}}\zeta(2s+1) = \beta(2s+1) + \frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\zeta(2s+1) + \frac{E_{2s}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]$$

where

$$\beta(2s+1) = -\frac{(2^{2s+1}-1)}{2^{2s+1}}\left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)}i\right]$$

and

$$\zeta(2s+1) = \frac{2^{2s+1}}{(2^{2s+1}-1)}\beta(2s+1) + \left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)} - \zeta(2s+1)i\right]i$$

is the closed-form expression for $\zeta(2n+1)$ such that

$$\frac{2^{2s+1}}{2^{2s+1}-1}\sum_{k=1}^{\infty}\frac{2}{(4k-1)^{s}} = \left[\frac{\mathbf{E_{2s}}(\pi i)^{2s+1}}{2(2^{2s+1}-1)\Gamma(2s+1)} - \zeta(2s+1)i\right]i$$

References

[1] Olver, Frank W. J., Lozier, Daniel W., Boisvert, Ronald F., Clark, Charles W. NIST Handbook of Mathematical Functions, Cambridge University Press (2010), ISBN 0521140633.

[2] Idowu, M. A. Fundamental relations between the Dirichlet beta function, Euler numbers, and Riemann zeta function for positive integers. Oct. 2012. arXiv:1210.5559.

[3] Idowu, M. A. Elegant expressions and generic formulas for the Riemann zeta function for integer arguments. Oct. 2012. arXiv:1210.5157.